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REVIEW

Solution of the Schrödinger equation for two q-deformed potentials by the SWKB method

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Abstract In this study, the supersymmetric Wentzel-Kramers-Brillouin (SWKB) approximation method has been discussed in detail. The entire bound state energy eigenvalues and the ground state wave function of the q-deformed potentials such as Pöschl-Teller and Rosen-Morse potentials have been obtained by SWKB approximation. We have shown that our results are in complete agreement with the results obtained by Eğrifes et al. (Physica Scripta 59:90–94, 1999; Physica Scripta 60:195–198, 1999).

Keywords Supersymmetric quantum mechanics · The lowest order SWKB quantization condition · q-deformed potentials

1 Introduction

The WKB method is one of the most useful approximation methods that is used for computing the energy eigenvalues of Hamiltonian of the Schrodinger equation. This method is exact in the case of only two potentials, i.e. one dimensional harmonic oscillator problem and the Morse potential. For other solvable potentials, the energy eigenvalues computed by this method are not exact [1]. This setback has been remedied after SWKB approximation that has been inspired by supersymmetric quantum mechanics. This approximation method is introduced for the first time by Comtet et al. [2]. All of the solvable potentials could be solved with this method which can be applied to any potential whose ground state ($n = 0$) wave function ψ_0 (and hence eigenvalue E_0) is known [3]. The purpose of this article is to describe the lowest order SWKB approximation method and give physical applications. The plan of the paper

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is the following. In Sect. 2, we review the main ideas of supersymmetric quantum mechanics basically and introduce the lowest order SWKB approximation method. In Sect. 3, we give a brief explanation about q-deformed hyperbolic functions and solve q-deformed Pöschl-Teller and q-deformed Rosen-Morse potential by using the lowest order SWKB approximation. Finally, in Sect. 4 we present a brief discussion of the results achieved.

2 Supersymmetric quantum mechanics and the lowest order SWKB approximation method

One of the important approaches in solving the Schrödinger equation is to factorize the Hamiltonian. Thus, operators A and $A^\dagger$ are introduced in supersymmetric quantum mechanics:

$$A = \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{1}$$

With the help of these operators Hamiltonians H_1 and H_2 are obtained as:

$$H_1 = A^\dagger A, \quad H_2 = AA^\dagger \tag{2}$$

$$H_1 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x), \tag{3}$$

$$H_2 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_2(x), \tag{4}$$

where

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x), \tag{5}$$

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \tag{6}$$

and $W'(x)$ stands for the derivative of $W(x)$.

In supersymmetric quantum mechanics; $W(x)$ and $V_{1,2}(x)$ are called superpotential and supersymmetric partner potentials, respectively [1]. If $V_1(x)$ is given, then the superpotential is obtained from Eq. 5. This equation is a Riccati differential equation. Moreover $W(x)$ can be obtained from the fact that operator A destroys the ground state wave function of Hamiltonian H_1,

$$A\psi_0^{(1)}(x) = 0. \tag{7}$$

Thus superpotential $W(x)$ is related to $\psi_0^{(1)}(x)$ by

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'^{(1)}(x)}{\psi_0^{(1)}(x)}, \tag{8}$$

$$\psi_0^{(1)}(x) = \exp\left(-\frac{\sqrt{2m}}{\hbar} \int^x W(x')\mathrm{d}x'\right). \tag{9}$$

For one dimensional problems, WKB approximation method gives

$$\int_{x_1}^{x_2} \mathrm{d}x \sqrt{2m[E_n - V(x)]} = (n + 1/2)\hbar\pi, \tag{10}$$

where the turning points x_1 and x_2 are given by $E_n = V(x_1) = V(x_2)$. For the potential $V_1(x)$ corresponding to the superpotential $W(x)$, the lowest order WKB quantization condition given in Eq. 10 is written as:

$$\int_{x_1}^{x_2} \sqrt{2m\left[E_n^{(1)} - W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x)\right]}\,\mathrm{d}x = (n + 1/2)\hbar\pi. \tag{11}$$

Then, the left hand side of this equation is expanded in powers of $\hbar$, in the limit $\hbar \to 0$:

$$\int_a^b \sqrt{2m\left[E_n^{(1)} - W^2(x)\right]}\,\mathrm{d}x = n\hbar\pi, \quad n = 0, 1, 2, \ldots, \tag{12}$$

where a and b are the turning points defined by $E_n^{(1)} = W^2(a) = W^2(b)$, and $\lim_{\hbar \to 0} x_{1,2} = a, b$ [3].

This is supersymmetric WKB (SWKB) quantization condition for Hamiltonian H_1. Proceeding in the same way the SWKB quantization condition for the potential $V_2(x)$ turns out to be;

$$\int_a^b \sqrt{2m\left[E_n^{(2)} - W^2(x)\right]}\,\mathrm{d}x = (n + 1)\hbar\pi, \quad n = 0, 1, 2, \ldots. \tag{13}$$

Some remarks are in order at this stage [3];

(1) For $n = 0$ the turning points a and b in Eq. 12 are coincident since $E_0^{(1)} = 0$.
(2) On comparing Eqs. 12 and 13 it follows that the lowest order SWKB quantization condition preserves the level degeneracy $E_{(n+1)}^{(1)} = E_n^{(2)}$.

For an exactly solvable potential $V(x)$ with n bound states and ground state energy E_0, the supersymmetric partner potential $V_1(x)$ is written as $V(x) - E_0$. Therefore the ground state energy is zero by construction [1]. Then this ground state energy and the energy $E_n^{(1)}$ that is obtained by the lowest order SWKB method are summed up:

$$E_{nq} = E_{nq}^{(1)} + E_{0q}^{(1)}. \tag{14}$$

In this way by starting from a solvable problem with n bound states, the energy eigenvalues can be obtained exactly.

3 *q*-deformed potentials

A new class of potentials which are called "deformed hyperbolic potentials" were recently introduced by Arai [4]. These potentials are defined in terms of "deformed hyperbolic functions" that are given by:

$$\cosh_q x = \frac{e^x + qe^{-x}}{2}, \quad \sinh_q x = \frac{e^x - qe^{-x}}{2}, \quad \tanh_q x = \frac{\sinh_q x}{\cosh_q x}, \tag{15}$$

where $q > 0$ is a real parameter. In this work; we intend to solve Schrödinger equation for two of these potentials, i.e. q-deformed Pöschl-Teller potential and q-deformed Rosen-Morse potential with SWKB method.

3.1 Solution of Schrödinger equation for q-deformed Pöschl-Teller potential with SWKB method

The deformation of the original Pöschl-Teller potential is:

$$V_q(x) = -\frac{V_0}{\cosh_q^2(\alpha x)} = -4V_0 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2}. \tag{16}$$

This potential arises in the study of solitons [6]. A plot of q-deformed Pöschl-Teller potential is presented in Fig. 1. With the aim of calculating the energy eigenvalues; we start with the superpotential introduced as $W(x) = Atanh_q(\alpha x)$, where A is a parameter. For this superpotential, the supersymmetric partner potential $V_1(x)$ is:

$$V_q(x) - E_{0q}^{(1)} = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x). \tag{17}$$

The ground state energy eigenvalue can be obtained from the solution of this equation resulting in:

$$E_{0q}^{(1)} = -\frac{\hbar^2\alpha^2}{8m}\left(-1 + \sqrt{1 + \frac{8mV_0}{q\hbar^2\alpha^2}}\right)^2. \tag{18}$$

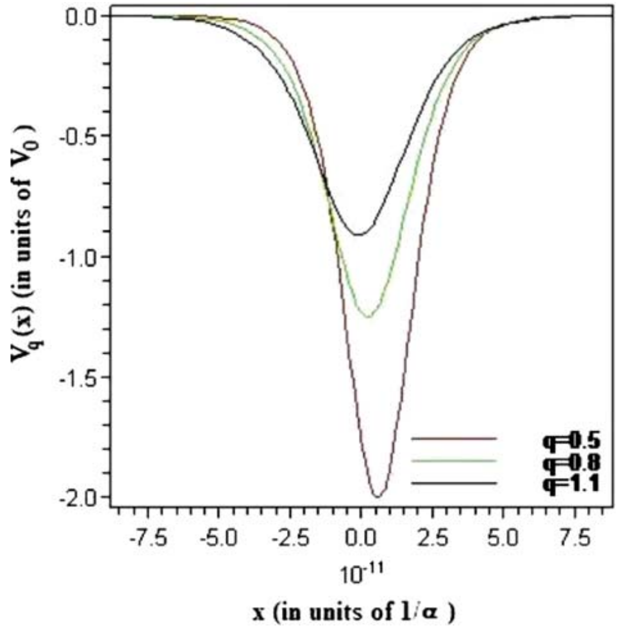

Fig. 1 q-deformed Pöschl-Teller potential for three different values of the deformation parameter q, where $\alpha = 2.75$ have been chosen

To find the bound state energy eigenvalues; the superpotential $W(x) = A\tanh_q(\alpha x)$ is substituted in SWKB quantization condition given in Eq. 12:

$$\int_{x_1}^{x_2} \sqrt{2m\left[E_{nq}^{(1)} - A^2\tanh_q^2(\alpha x)\right]}\,\mathrm{d}x = n\hbar\pi, \quad n = 0, 1, 2, \ldots. \tag{19}$$

With a change of variable $y = \tanh_q(\alpha x)$ the SWKB integral now reads:

$$\frac{A\sqrt{2m}}{\alpha}\int_{y_1}^{y_2} \frac{\mathrm{d}y}{(1-y^2)}\sqrt{\left(y + \frac{\sqrt{E_{nq}^{(1)}}}{A}\right)\left(\frac{\sqrt{E_{nq}^{(1)}}}{A} - y\right)} = n\hbar\pi. \tag{20}$$

The turning points are the roots of the equation:

$$A\tanh_q(\alpha x) = Ay = \pm\sqrt{E_{nq}^{(1)}} \tag{21}$$

and are calculated as:

$$y_1 = a = -\frac{\sqrt{E_{nq}^{(1)}}}{A}, \quad y_2 = b = \frac{\sqrt{E_{nq}^{(1)}}}{A}. \tag{22}$$

To find the solution of the integral in Eq. 20, we can use the integral equation given in [5]:

$$\int_a^b \frac{dy}{(1-y^2)}\sqrt{(y-a)(b-y)} = \frac{\pi}{2}[2-\sqrt{(1-a)(1-b)}-\sqrt{(1+a)(1+b)}]. \tag{23}$$

Then the energy levels $E_{nq}^{(1)}$ is achieved as follows:

$$E_{nq}^{(1)} = -\frac{2n\hbar\alpha}{\sqrt{2m}}\left[\frac{\hbar\alpha}{\sqrt{8m}}\left(-1+\sqrt{1+\frac{8mV_0}{q\hbar^2\alpha^2}}\right)\right]-\frac{n^2\hbar^2\alpha^2}{2m}. \tag{24}$$

After substituting Eqs. 18 and 24 in Eq. 14, the exact entire bound state energy eigenvalue spectra is obtained as:

$$E_{nq} = -\frac{\hbar^2\alpha^2}{8m}\left[-(2n+1)+\sqrt{1+\frac{8mV_0}{q\hbar^2\alpha^2}}\right]^2. \tag{25}$$

We obtain the ground state eigenfunction from Eq. 9 for this superpotential giving:

$$\psi_{0q}^{(1)}(x) = [\mathrm{sech}_q(\alpha x)]^{\frac{1}{2}\left(-1+\sqrt{1+\frac{8mV_0}{q\hbar^2\alpha^2}}\right)}. \tag{26}$$

3.2 Solution of Schrödinger equation for q-deformed Rosen-Morse potential with SWKB method

The deformation configuration of the original Rosen-Morse potential which gives agreeable results for molecular interactions [7], is:

$$V_q(x) = B_0 \tanh_q(\alpha x) - \frac{U_0}{\cosh_q^2(\alpha x)}. \tag{27}$$

A plot of q-deformed Rosen-Morse potential is presented in Fig. 2.

For this potential, proceeding in the same way that is used to solve the q deformed Pöschl-Teller potential, the superpotential given in terms of parameters A and B, is introduced as $W(x) = A\tanh_q(\alpha x) + \frac{B}{A}$. This superpotential is substituted in Eq. 5

$$V_q(x) - E_{0q}^{(1)} = 2B\tanh_q(\alpha x) - \left(A^2 + \frac{\hbar}{\sqrt{2m}}A\alpha\right)q\,\mathrm{sech}_q^2(\alpha x) + A^2 + \frac{B^2}{A^2}. \tag{28}$$

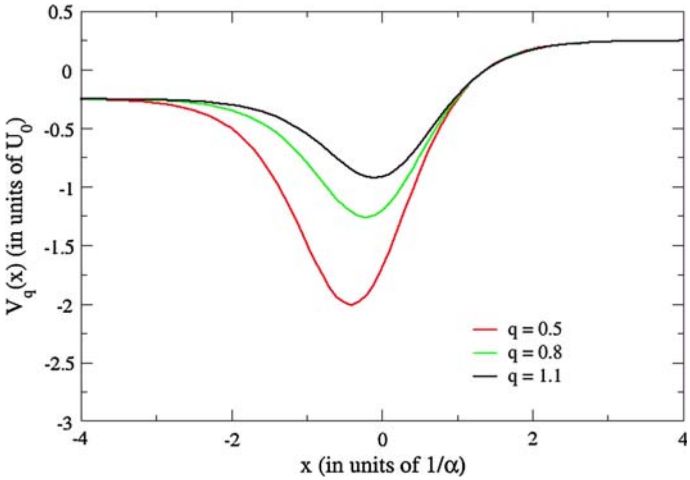

Fig. 2 q-deformed Rosen-Morse potential for three values of the deformation parameter q, where $\alpha = 1.0$ and $B_0 = \frac{U_0}{4}$ have been chosen

The solution of this equation leads to ground state energy eigenvalue given by:

$$E_{0q}^{(1)} = -\frac{\hbar^2\alpha^2}{8m}\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)^2 - \frac{\frac{2mB_0^2}{\hbar^2\alpha^2}}{\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)^2}. \tag{29}$$

With this superpotential Eq. 12 now becomes:

$$\int_{x_1}^{x_2}\sqrt{2m\left[E_{nq}^{(1)} - A^2\tanh_q^2(\alpha x) - 2B\tanh_q^2(\alpha x) - \frac{B^2}{A^2}\right]}\,\mathrm{d}x = n\hbar\pi,$$
$$n = 0, 1, 2, \ldots. \tag{30}$$

With a change of variable $y = \tanh_q(\alpha x)$ the SWKB integral is:

$$\frac{A\sqrt{2m}}{\alpha}\int_{y_1}^{y_2}\frac{\mathrm{d}y}{(1-y^2)}\sqrt{\left[-y^2 - \frac{\sqrt{2B}}{A^2}y + \frac{\sqrt{E_{nq}^{(1)}}}{A^2} - \frac{B^2}{A^4}\right]} = n\hbar\pi. \tag{31}$$

The turning points for this case are:

$$y_1 = a = -\frac{\sqrt{E_{nq}^{(1)}}}{A} - \frac{B}{A^2}\,,\; y_2 = b = \frac{\sqrt{E_{nq}^{(1)}}}{A} - \frac{B}{A^2}\,. \tag{32}$$

Using once more Eq. 23 for computing the integral and following the previous steps one now gets:

$$E_{nq}^{(1)} = \left[\frac{\hbar\alpha}{\sqrt{8m}}\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)\right]^2 - \left[\frac{\hbar\alpha}{\sqrt{8m}}\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)-\frac{n\hbar\alpha}{\sqrt{2m}}\right]^2 + \frac{B_0^2}{4\left[\frac{\hbar\alpha}{\sqrt{8m}}\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)\right]^2} - \frac{B_0^2}{4\left[\frac{\hbar\alpha}{\sqrt{8m}}\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)-\frac{n\hbar\alpha}{\sqrt{2m}}\right]^2}. \tag{33}$$

Moreover entire bound state energy eigenvalues are achieved as:

$$E_{nq} = -\frac{2\hbar^2\alpha^2}{m}\left[\frac{1}{16}\left(-(2n+1)+\sqrt{1+\frac{8mV_0}{q\hbar^2\alpha^2}}\right)^2 + \frac{m^2B_0^2}{\hbar^4\alpha^4}\frac{1}{\left(-(2n+1)+\sqrt{1+\frac{8mV_0}{q\hbar^2\alpha^2}}\right)^2}\right]. \tag{34}$$

The ground state energy eigenfunction is determined from Eq. 9 for this superpotential. The result is the following:

$$\psi_{0q}^{(1)}(x) = [\mathrm{sech}_q(\alpha x)]^{\frac{1}{2}\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)} \times \exp\left[-\frac{2m}{\hbar^2\alpha}\frac{B_0}{\left(-1+\sqrt{1+\frac{8\,mU_0}{q\hbar^2\alpha^2}}\right)}x\right]. \tag{35}$$

4 Conclusion

In this study, a brief description of the lowest order SWKB approximation method is given. With the purpose of the illustration of the method, we have solved Schrödinger equation for two q-deformed hyperbolic potentials, namely q-deformed Pöschl-Teller and q-deformed Rosen-Morse potentials and obtained their eigenvalue spectra analitically in addition to the ground state eigenfunctions. We have to emphasize that; when compared to some other methods namely Nikiforov–Uvarov method used by other authors [6,7] this method is favourable in the solution of Schrödinger equation for these potentials since it is based on a basic straightforward algebra.

References

1. A. Khare, Supersymmetry in Quantum Mechanics (2004). arXiv:math-ph/0409003v1
2. A. Comtet, A. Bandrauk, D.K. Campbell, Exactness of semiclassical bound state energies for supersymmetric quantum mechanics. Phys. Lett. **B150**, 159 (1985)
3. R. Dutt, A. Khare, U.P. Sukhatme, Supersymmetry-inspired WKB approximation in quantum mechanics. Am. J. Phys. **59**(8) (1991)
4. A. Arai, Exactly solvable supersymmetric quantum mechanics. J. Math. Anal. Appl. **158**, 63 (1991)
5. M. Hruska, W.Y. Keung, U. Sukhatme, Accuracy of Semiclassical Methods for Shape Invariant Potentials (1996). arXiv:quant-ph/9611030v1
6. H. Eğrifes, D. Demirhan, F. Fevzi Büyükkılıç, Polynominal solutions of the Schrödinger equation for the "deformed" hyperbolic potentials by Nikiforov–Uvarov method. Physica Scripta. **59**, 90–94 (1999)
7. H. Eğrifes, D. Demirhan, F. Fevzi Büyükkılıç, Exact solutions of the Schrödinger equation for two deformed hyperbolic molecular potentials. Physica Scripta. **60**, 195–198 (1999)